Final Exam — Analysis (WPMA14004)

Tuesday 28 January 2020, 8.30h–11.30h

University of Groningen

Instructions

1. The use of calculators, books, or notes is not allowed.
2. Provide clear arguments for all your answers: only answering "yes", "no", or "42" is not sufficient. You may use all theorems and statements in the book, but you should clearly indicate which of them you are using.
3. The total score for all questions equals 90. If p is the number of marks then the exam grade is $G = 1 + p/10$.

Problem 1 (3 + 10 + 2 = 15 points)

Consider the set

$$A = \left\{ \frac{10x + 7}{5x + 3} \; : \; x \geq 0 \right\}.$$

(a) Show that 2 is a lower bound for the set A.

(b) Prove that $\inf A = 2$.

(c) Is $\inf A$ an element of A?

Problem 2 (4 + 7 + 4 = 15 points)

Consider the sequence (s_n) defined by $s_n = \sum_{k=1}^{n} \frac{k}{3^k}$.

(a) It is given that $k/3^k < 1/2^k$ for all $k \in \mathbb{N}$. Use this to prove that (s_n) converges.

(b) Prove that $s_{n+1} = \frac{1}{3} s_n + \sum_{k=1}^{n+1} \frac{1}{3^k}$ for all $n \in \mathbb{N}$.

(c) Compute the value of the series $\sum_{k=1}^{\infty} \frac{k}{3^k}$.

Problem 3 (5 + 10 = 15 points)

Let $K \subseteq \mathbb{R}$ be compact, and consider the following set:

$$A = \bigcup_{x \in K} [x - 1, x + 1].$$

(a) Show that A is bounded.

(b) Show that A is compact by proving that A is also closed.

Problem 4 (7 + 4 + 4 = 15 points)

Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Let $a < b$ and assume that

$$\big[f(b) - f(a)\big] f'(b) < 0.$$

Prove the following statements:

(a) The following function is continuous on $[a, b]$:

$$g(x) = \begin{cases} \dfrac{f(b) - f(x)}{b - x} & \text{if } a \le x < b, \\[2ex] f'(b) & \text{if } x = b. \end{cases}$$

(b) There exists $s \in (a, b)$ such that $g(s) = 0$.

(c) There exists $t \in (s, b)$ such that $f'(t) = 0$.

Problem 5 (6 + 6 + 3 = 15 points)

(a) Show that for any $x > 0$ and $n \in \mathbb{N}$ there exists $c \in (0, x)$ such that

$$\ln(1 + x) - \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} x^k = \frac{(-1)^n}{(n + 1)(1 + c)^{n+1}}\, x^{n+1}.$$

(b) Prove that the series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k$$

converges uniformly to $\ln(1 + x)$ on the interval $[0, 1]$.

(c) Let $a = \frac{1}{2} - \frac{1}{8} + \frac{1}{24} = \frac{5}{12}$. Show that $|\ln(\frac{3}{2}) - a| < \frac{1}{64}$.

Problem 6 (3 + 12 = 15 points)

Consider the function $f(x) = 1/(1 + x^2)$.

(a) Argue that f is integrable on $[0, 1]$ *without* using upper or lower sums.

(b) Let $n \in \mathbb{N}$. Use the partition $P = \big\{k/n \,:\, k = 0, 1, \ldots, n\big\}$ of $[0, 1]$ to show that

$$\sum_{k=1}^{n} \frac{4n}{n^2 + k^2} \le \pi \le \sum_{k=1}^{n} \frac{4n}{n^2 + (k - 1)^2}.$$

End of test (90 points)

Solution of Problem 1 (3 + 10 + 2 = 15 points)

(a) *Method 1.* For all $x \geq 0$ we have

$$\frac{10x+7}{5x+3} > \frac{10x+6}{5x+3} = 2,$$

which means that 2 is a lower bound for the set A.
(3 points)

Method 2. For all $x \geq 0$ we have

$$\frac{10x+7}{5x+3} = \frac{10x+6}{5x+3} + \frac{1}{5x+3} = 2 + \frac{1}{5x+3} > 2,$$

which means that 2 is a lower bound for the set A.
(3 points)

Method 3. For all $x \geq 0$ we have the following equivalences:

$$\frac{10x+7}{5x+3} \geq 2 \quad \Leftrightarrow \quad 10x+7 \geq 10x+6 \quad \Leftrightarrow \quad 7 \geq 6,$$

and the latter inequality is obviously true. Therefore, the first inequality holds as well, which shows that 2 is a lower bound for the set A.
(3 points)

(b) *Method 1.* If $\ell \in \mathbb{R}$ is any lower bound for A, then

$$\ell \leq \frac{10x+7}{5x+3} \quad \text{for all} \quad x \geq 0,$$

or, equivalently,

$$(5\ell - 10)x \leq 7 - 3\ell \quad \text{for all} \quad x \geq 0.$$

(3 points)

If $\ell > 2$, then $5\ell - 10 > 0$ and $7 - 3\ell < 1$, which implies that the inequality above cannot hold for all $x \geq 0$. Therefore, we conclude that $\ell \leq 2$, which implies that $\inf A = 2$ by definition.
(7 points)

Method 2. Let $\epsilon > 0$ be arbitrary, and take $x > 1/5\epsilon - 3/5$. Then it follows that

$$\frac{10x+7}{5x+3} = 2 + \frac{1}{5x+3} < 2 + \epsilon,$$

which means that $2+\epsilon$ is not a lower bound for A. Since $\epsilon > 0$ is arbitrary it follows from Lemma 1.3.8 that $\inf A = 2$.
(10 points)

(c) *Method 1.* $2 \notin A$ since all elements in A are strictly greater than 2.
(2 points)

Method 1. If $2 \in A$, then there exists $x \geq 0$ such that

$$\frac{10x+7}{5x+3} = 2,$$

or, equivalently, $7 = 6$ which is obviously not true. Hence, $2 \notin A$.
(2 points)

Solution of Problem 2 (4 + 7 + 4 = 15 points)

(a) *Method 1.* Clearly, the sequence (s_n) is increasing as $s_{n+1} - s_n = (n+1)/3^{n+1} > 0$ so that $s_n < s_{n+1}$ for all $n \in \mathbb{N}$.
(1 point)

Using the give inequality shows that

$$s_n = \sum_{k=1}^{n} \frac{k}{3^k} < \sum_{k=1}^{n} \frac{1}{2^k} < \sum_{k=1}^{\infty} \frac{1}{2^k} = 1,$$

for all $n \in \mathbb{N}$. Therefore, the sequence (s_n) is also bounded.
(2 points)

By the Monotone Convergence Theorem it follows that (s_n) is convergent.
(1 point)

Method 2. Note that the series

$$\sum_{k=1}^{\infty} \frac{1}{2^k}$$

is a convergent geometric series. (The fact that the series starts at $k = 1$ instead of $k = 0$ is not relevant for convergence issues. However, it *is* relevant for the value of the sum!).
(2 points)

With the given inequality we can apply the Comparison Test to conclude that the series

$$\sum_{k=1}^{\infty} \frac{k}{3^k}$$

also converges. Since (s_n) is the sequence of partial sums for this series, it follows that (s_n) converges.
(2 points)

(b) *Method 1.* On the one hand we have

$$s_2 = \frac{1}{3} + \frac{2}{9}.$$

On the other hand we have

$$\frac{1}{3}s_1 + \sum_{k=1}^{2} \frac{1}{3^k} = \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} + \frac{1}{9} = \frac{1}{3} + \frac{2}{9}.$$

Therefore, the formula holds for $n = 1$.
(1 point)

Now assume the formula holds for *some* $n \in \mathbb{N}$. Then it follows that

$$
\begin{aligned}
s_{n+2} &= \sum_{k=1}^{n+2} \frac{k}{3^k} \\
&= s_{n+1} + \frac{n+2}{3^{n+2}} \\
&= \frac{1}{3}s_n + \sum_{k=1}^{n+1} \frac{1}{3^k} + \frac{n+2}{3^{n+2}} && \text{(by induction hypothesis)} \\
&= \frac{1}{3}s_n + \frac{1}{3} \cdot \frac{n+1}{3^{n+1}} + \sum_{k=1}^{n+1} \frac{1}{3^k} + \frac{1}{3^{n+2}} \\
&= \frac{1}{3}\left(s_n + \frac{n+1}{3^{n+1}}\right) + \sum_{k=1}^{n+1} \frac{1}{3^k} + \frac{1}{3^{n+2}} = \frac{1}{3}s_{n+1} + \sum_{k=1}^{n+2} \frac{1}{3^k},
\end{aligned}
$$

which shows that the formula is also true for $n+1$. By induction the formula holds for all $n \in \mathbb{N}$.
(6 points)

Method 2. We can also prove the formula without induction as follows:

$$
\begin{aligned}
s_{n+1} &= \sum_{k=1}^{n+1} \frac{k}{3^k} \\
&= \frac{1}{3} + \sum_{k=2}^{n+1} \frac{k}{3^k} \\
&= \frac{1}{3} + \frac{1}{3}\sum_{k=2}^{n+1} \frac{k}{3^{k-1}} \\
&= \frac{1}{3} + \frac{1}{3}\sum_{k=1}^{n} \frac{k+1}{3^k} \\
&= \frac{1}{3}\sum_{k=1}^{n} \frac{k}{3^k} + \frac{1}{3} + \frac{1}{3}\sum_{k=1}^{n} \frac{1}{3^k} = \frac{1}{3}s_n + \sum_{k=1}^{n+1} \frac{1}{3^k}.
\end{aligned}
$$

(7 points)

(c) From part (a) we know that $s = \lim s_n$ exists. Taking the limit $n \to \infty$ in both left and right hand side of the formula derived in part (b) gives

$$s = \frac{1}{3}s + \sum_{k=1}^{\infty} \frac{1}{3^k},$$

or, equivalently,

$$s = \frac{3}{2}\sum_{k=1}^{\infty} \frac{1}{3^k}.$$

(2 points)

The sum formula of the geometric series (note that the series starts at $k=1$ instead of $k=0$) gives

$$s = \frac{3}{2}\left(\frac{1}{1-1/3} - 1\right) = \frac{3}{4}.$$

(2 points; 1 point when s not correct)

Solution of Problem 3 (5 + 10 = 15 points)

(a) Since K is compact it follows that K is bounded. This means that there exists a constant $M \geq 0$ such that

$$|x| \leq M \quad \text{for all} \quad x \in K,$$

or, equivalently,

$$-M \leq x \leq M \quad \text{for all} \quad x \in K,$$

(1 point)

Let $a \in A$ be arbitrary. Then there exists $x \in K$ such that $a \in [x-1, x+1]$, or equivalently,

$$x - 1 \leq a \leq x + 1.$$

Since $-M - 1 \leq x - 1$ and $x + 1 \leq M$ we obtain

$$-M - 1 \leq a \leq M + 1$$

so that $|a| \leq M + 1$. We conclude that A is bounded.
(4 points)

(b) Assume that y is a limit point of A. Then there exists a convergent sequence (a_n) in A such that $a_n \to y$ and $a_n \neq y$ for all $n \in \mathbb{N}$.
(2 points)

Since the sequence (a_n) is contained in A there exists a sequence (x_n) in K such that

$$x_n - 1 \leq a_n \leq x_n + 1 \quad \text{for all} \quad n \in \mathbb{N}.$$

(2 points)

Since K is compact, the sequence (x_n) has a convergent subsequence (x_{n_k}) such that $x_{n_k} \to x$ with $x \in K$. We have

$$x_{n_k} - 1 \leq a_{n_k} \leq x_{n_k} + 1 \quad \text{for all} \quad k \in \mathbb{N}.$$

Taking the limit $k \to \infty$ and using the Order Limit Theorem gives

$$x - 1 \leq y \leq x + 1.$$

Since $x \in K$ it follows that $y \in A$. We conclude that A is closed.
(5 points)

Since A is bounded and closed it follows that A is compact.
(1 point)

Solution of Problem 4 (7 + 4 + 4 = 15 points)

(a) Since f is differentiable it is also continuous.
(1 point)

On the interval $[a, b)$ the function g is a quotient of two continuous. The Algebraic Continuity Theorem implies that g is continuous on $[a, b)$.
(3 points)

Since f is differentiable in $x = b$ we have that

$$\lim_{x \to b} g(x) = \lim_{x \to b} \frac{f(b) - f(x)}{x - b} = f'(b) = g(b),$$

which shows that g is continuous at $x = b$.
(3 points)

(b) We have that

$$g(a) = \frac{f(b) - f(a)}{b - a} \quad \text{and} \quad g(b) = f'(b).$$

It is given that $f(b) - f(a)$ and $f'(b)$ have opposite sign. Since $b - a > 0$ it the follows that $g(a)$ and $g(b)$ have opposite sign. By the Intermediate Value Theorem it follows that there exists $s \in (a, b)$ such that $g(s) = 0$.
(4 points)

(c) By part (b) we have that $g(s) = 0$ from which it follows that $f(s) = f(b)$. Now either apply the Mean Value Theorem or Rolle's Theorem on the interval $[s, b]$ to conclude that there exists $t \in (s, b)$ such that $f'(t) = 0$.
(4 points)

Solution of Problem 5 (6 + 6 + 3 = 15 points)

(a) For $f(x) = \ln(1+x)$ we have

$$f^{(1)}(x) = (1{+}x)^{-1}, \quad f^{(2)}(x) = -(1{+}x)^{-2}, \quad f^{(3)}(x) = 2(1{+}x)^{-3}, \quad f^{(4)}(x) = -6(1{+}x)^{-4}.$$

More generally, for any $n \in \mathbb{N}$ we have

$$f^{(n)}(x) = \frac{(-1)^{n+1}\,(n-1)!}{(1+x)^n}.$$

(3 points; induction not needed, just observing the pattern suffices)

Then we have the polynomial

$$s_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}x^k$$

If $x > 0$, then Lagrange's Remainder Theorem implies that there exists $c \in (0, x)$ such that

$$E_n(x) = f_n(x) - s_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$$

Plugging in our formula the derivatives of f gives

$$E_n(x) = \frac{(-1)^{n+2}\,n!}{(n+1)!\,(1+c)^{n+1}}x^{n+1} = \frac{(-1)^n}{(n+1)(1+c)^{n+1}}x^{n+1},$$

which proves the requested formula.
(3 points)

(b) For all $x \in [0, 1]$ we have that

$$|E_n(x)| = \frac{1}{(n+1)(1+c)^{n+1}}|x|^{n+1} \leq \frac{1}{(n+1)(1+c)^{n+1}} < \frac{1}{n+1}.$$

Therefore, we obtain that

$$\sup_{x\in[0,1]} |f_n(x) - s_n(x)| \leq \frac{1}{n+1},$$

which immediately implies that

$$\lim \left(\sup_{x\in[0,1]} |f_n(x) - s_n(x)| \right) = 0.$$

This proves that the sequence (s_n) converges uniformly to f on the interval $[0, 1]$.
(6 points)

(c) With $n = 3$ and $x = \frac{1}{2}$ it follows from part (a) that there exists $c \in (0, \frac{1}{2})$ such that

$$\ln(\tfrac{3}{2}) - a = -\frac{1}{64(1+c)^4}.$$

Taking the absolute value of both sides gives

$$|\ln(\tfrac{3}{2}) - a| = \frac{1}{64(1+c)^4} < \frac{1}{64}.$$

(3 points)

Solution of Problem 6 (3 + 12 = 15 points)

(a) *Method 1.* The function f is continuous on $[0,1]$ and therefore integrable.
(3 points)

Method 2. The function f is decreasing on $[0,1]$ and therefore integrable.
(3 points)

(b) Since f is decreasing on $[0,1]$, the upper sum of f with respect to P is given by

$$\begin{aligned} U(f,P) &= \sum_{k=1}^{n} M_k(x_k - x_{k-1}) \qquad\qquad M_k = \sup\{f(x) \,:\, x \in [x_{k-1}, x_k]\} \\ &= \sum_{k=1}^{n} f(x_{k-1})(x_k - x_{k-1}) \\ &= \sum_{k=1}^{n} \frac{1}{1+((k-1)/n)^2} \cdot \left(\frac{k}{n} - \frac{k-1}{n}\right) \\ &= \sum_{k=1}^{n} \frac{1}{1+((k-1)/n)^2} \cdot \frac{1}{n} \\ &= \sum_{k=1}^{n} \frac{n}{n^2+(k-1)^2}. \end{aligned}$$

(4 points)

The lower sum of f with respect to P is given by

$$\begin{aligned} L(f,P) &= \sum_{k=1}^{n} m_k(x_k - x_{k-1}) \qquad\qquad m_k = \inf\{f(x) \,:\, x \in [x_{k-1}, x_k]\} \\ &= \sum_{k=1}^{n} f(x_k)(x_k - x_{k-1}) \\ &= \sum_{k=1}^{n} \frac{1}{1+(k/n)^2} \cdot \left(\frac{k}{n} - \frac{k-1}{n}\right) \\ &= \sum_{k=1}^{n} \frac{1}{1+(k/n)^2} \cdot \frac{1}{n} \\ &= \sum_{k=1}^{n} \frac{n}{n^2+k^2}. \end{aligned}$$

(4 points)

For all partitions P of $[0,1]$ we have the inequality

$$L(f,P) \leq \int_0^1 f(x)\,dx \leq U(f,P).$$

(2 points)

Finally, the Fundamental Theorem of Calculus gives

$$\int_0^1 \frac{1}{1+x^2}\,dx = \big[\arctan(x)\big]_0^1 = \arctan(1) = \frac{\pi}{4},$$

which implies the requested inequality.
(2 points)